LINEAR PROGRAMMING UNDER UNCERTAINTY IN AN L_∞ SPACE*

BY
CHARLES S. FISHER

TECHNICAL REPORT NO. 7
DECEMBER, 1962

PREPARED UNDER CONTRACT Nonr-222 (77)
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Introduction

The mathematical problem discussed here is encountered in an attempt to generalize a class of economic models based on the minimax idea in game theory. The particular situation considered is that of "linear programming under uncertainty." In brief a probability space $(\Omega, \mathcal{A}, P)$ is given and also a finite family $\mathcal{B}_i$, $i=1,\ldots,n$, of sub-σ-fields of $\mathcal{A}$, and a linear continuous functional a on the space of n-tuples $e = (e_1,\ldots,e_n)$ where the e_i are real-valued functions on Ω, measurable with respect to $\mathcal{B}_i$. The problem then is to characterize a maximum of $\langle e,a \rangle$ when e is constrained to lie in some convex set.

The solution to this problem when Ω is finite and the constraints are given by matrix inequalities is given by R. Radner in [6]. In his paper an economic interpretation of the model is discussed. What follows is an extension to the situation in which Ω is a nonfinite, countable set, say the positive integers, $\mathcal{B}_i$ sub-σ-fields of the discrete σ-field, and the constraints are of the form $eT \leq b$ where T is an n x n matrix whose entries t_{ij} are functions on Ω, and $\leqq$ is the usual ordering on Euclidian n- or m-space, and b is

an m-tuple of functions on Ω. The linear functional to be maximized is of the form

$$\mathcal{E}(e\cdot a) = \int_{\Omega} \sum_{i=1}^{n} e_i(\omega)a_i(\omega)dP(\omega) \ .$$

As when working with linear programming problems in finite dimensional spaces, we will characterize the dual program and establish sufficient conditions for the equivalence of the existence of a maximum and that of a nonnegative saddle point of the associated Lagrangian form.

The question as to what topology is appropriate does not arise when the function spaces are finite dimensional (i.e., Ω is a finite set) because in this situation the spaces are reflexive and their weak and strong topologies are identical. Whereas the theorems for separation of convex sets in infinite dimensional locally convex linear topological spaces depend on the class of linear functionals given and the topology in which the sets are closed. For discussions of this see Hurwicz [3] and [2], chapter 5.

The topology that will be used is that of an L_∞ norm on an n-tuple of functions and instead of considering the entire conjugate of L_∞ as the setting for the dual problem, we have taken L_1. This restriction is imposed because the dual of L_∞ has no natural economic analogue. The presentation in what follows is for the case of couples of functions. Its extension to the n x m case is immediate.

There are various different ways of stating and proving duality theorems. In what follows, a variation of D. Bratton's

excellent, though unfortunately unpublished, duality theorem in [1] is presented. The treatment of the problem is also analogous to that done by Kretchmer in [4] but simplified because of the special properties of the spaces considered.

Preliminary Lemmas and an Example of Duality

Let $\Omega = \{1,2,3,\ldots\}$, $\mathcal{A}$ = discrete σ-field, $\mathcal{B}_i$, $i=1,2$ sub-σ-fields of $\mathcal{A}$ and P a positive probability measure on $\mathcal{A}$ such that $P(\Omega) = 1$ and $P(\omega) > 0$ for all $\omega \in \Omega$.

Also let $E = L_\infty(\Omega,\mathcal{B}_1,P) \times L_\infty(\Omega,\mathcal{B}_2,P)$, $F = L_\infty(\Omega,\mathcal{A}P) \times L_\infty(\Omega,\mathcal{A}P)$, and $E' = L_1(\Omega,\mathcal{B}_1,P) \times L_1(\Omega,\mathcal{B}_2,P)$, $F' = L_1(\Omega,\mathcal{A},P) \times L_1(\Omega,\mathcal{A},P)$. [E = conjugate of E' in its norm topology ($\|(e'^1,e'^2)\| = \max[\|e'^2\|,\|e'^2\|]$) and similarly F = conjugate of F'.] If $T = \begin{pmatrix} t_{11}(\omega), t_{12}(\omega) \\ t_{21}(\omega), t_{22}(\omega) \end{pmatrix}$ is a linear transformation defined on E by $(e^1,e^2)\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$ then eT belongs to F and T is bounded (i.e., continuous) if $\sup\limits_{i,j,\Omega} |t_{ij}(\omega)| = M < \infty$. For if $eT = f$ then $\|f^1\| = \sup\limits_{\omega} |e^1(\omega)t_{11}(\omega) + e^2(\omega)t_{21}(\omega)| \leqq M2\|e\|$. The same holds for f^2, therefore $\|eT\| \leqq 2M\|e\|$.

The set $P_E = \{e \in E\colon (e^1(\omega),e^2(\omega)) \geqq (0,0) \text{ for all } \omega \in \Omega\}$ will be called the positive cone of E. Its conjugate is $P_E^{\oplus} = \{e' \in E'\colon \langle e,e' \rangle \geqq 0\}$. By taking ones and zero's in the appropriate coordinates, we have that $P_E^{\oplus} = \{e'\colon (e'^1,e'^2) \geqq (0,0) \text{ for all } \omega \in \Omega\}$. Similarly $P_F = \{f\colon f \geqq 0\}$ and $P_F^{\oplus} = \{f'\colon \langle f,f' \rangle \geqq 0\} = \{f'\colon f' \geqq 0 \text{ (i.e., } (f^1,f^2) \geqq (0,0))\}$. Also define $P_R = \{\lambda\colon \lambda \text{ real number and } \lambda \geqq 0\}$.

We will consider the following programming problem.

Given $a \in E'$ and $b \in F$, then under what conditions is the existence of a maximum of $\langle e,a \rangle$ subject to the constraints $e \geqq 0$ and $eT \leqq b$ (i.e., $b-eT \in P_F$) equivalent to the existence of a positive saddle point of the Lagrangian, $\Phi(e,f') = \langle e,a \rangle + \langle b-eT,f' \rangle$.

To do this we will first prove a specialization of the Minkowski-Farkas lemma and its consequences to D. Bratton's version of the duality theorem as it appears in Cowles Commission Discussion Paper, Mathematics No. 427 [1].

The reason that the following discussion is necessary is that the duality theorem depends heavily on the type of closure of certain convex sets. If X is a Banach space X^*, X^{**} is first and second conjugates then in general the X-topology (weak $*$) of X^*, the X^{**}-topology (weak), and the norm topology of X^* are distinct and of increasing fineness (i.e., contain more open sets). We will consider L_∞ whose conjugate space strictly includes L_1; and L_1 in the case with which we are dealing is the conjugate of a Banach space and so possesses a weak $*$-topology.

The situation will be made clear by comparing the following two lemmas to the Minkowski-Farkas lemma.

__Lemma__ 1. <u>Let</u> $X = L_1(A,\mathcal{A},\mu)$ <u>and</u> $Y = L_\infty(A,\mathcal{A},\mu)$ <u>where</u> μ <u>is finite</u>. <u>If</u> P <u>is a convex cone</u> $\subseteq X$ <u>then</u> $P^{\oplus\oplus}$ <u>coincides with the</u> L_∞ <u>closure of</u> P <u>in</u> L_1.

<u>Proof</u>:

$$P^{\oplus} = \{y \in Y: \langle y,p \rangle \geqq 0 \text{ for all } p \in P\}$$

$$P^{\oplus\oplus} = \{x \in X: \langle y,x \rangle \geqq 0 \text{ for all } y \in P^{\oplus}\} \quad .$$

1) L_∞ closure of $P \subseteq P^{\oplus\oplus}$. If $z \in L_\infty$ closure of P, then there exists a net $P_\alpha \in P$ such that for all $w \in Y$ $\langle w,p_\alpha\rangle \to \langle w,z\rangle$ hence for all $y \in P^{\oplus}$ $\langle y,p_\alpha\rangle \to \langle y,z\rangle$ but $\langle y,p_\alpha\rangle \geqq 0$, therefore $\langle y,z\rangle \geqq 0$ and $z \in P^{\oplus\oplus}$.

2) $P^{\oplus\oplus} \subseteq L_\infty$ closure of P. If $x_o \notin L_\infty$ closure of P, then since the L_∞ closure of P is convex and Y equals the set of all continuous linear functions on X with the L_∞ topology ([2], p. 421), we have by the Mazur-Bourgin theorem ([2], p. 417) that there exists a $y_o \in Y$ such that $y_o(x_o) < c-\epsilon < c \leqq y_o(P)$ but P is a cone so $y_o(P) \geqq 0$; hence $y_o \in P^{\oplus}$ and $y_o(x_o) < 0$; therefore $x_o \notin P^{\oplus\oplus}$. And the lemma is established. L_∞ closure of $P = P^{\oplus\oplus}$.

On reversing the roles of X and Y in Lemma 1, we get by the same argument that the L_1 closure of P equals $P^{\oplus\oplus}$.

Note that in the above, because P is convex, its weak closure and closure in the norm coincide. In the L_1 spaces that we will consider, Ω is countable and $P(\omega) > 0$ for all $\omega \in \Omega$ so that the weak and norm topologies are the same ([2], p. 295).

<u>Lemma</u> 2. <u>Let</u> E, E', F, F', <u>and</u> T <u>be</u> <u>as</u> <u>given</u> <u>above</u> $T: E \to F$ <u>then</u> 1) <u>the</u> <u>transpose</u> T' <u>of</u> T <u>where</u> $T': F' \to E'$ <u>is</u> <u>a</u> <u>well-defined</u>, <u>linear</u>, <u>continuous</u> <u>transformation</u> <u>such</u> <u>that</u> $\langle eT,f'\rangle = \langle e,T'_f\rangle$ <u>for</u> $e \in E$, $f' \in F'$; <u>and</u> 2) <u>if</u> Q <u>is</u> <u>a</u> <u>closed</u> [<u>weak</u> <u>or</u> <u>norm</u> <u>closed</u>] <u>convex</u> <u>cone</u> <u>in</u> F, <u>then</u> $[T^{-1}(Q)]^{\oplus}$ = E <u>closure</u> <u>of</u> $T'(Q^{\oplus})$ <u>in</u> E'.

Proof:

1) For $e \in E$, $f' \in F'$,

$$\langle eT, f' \rangle = \int_\Omega [e^1 t_{11} + e^2 t_{21}] f'^1 dP + \int_\Omega [e^1 t_{12} + e^2 t_{22}] f^{2'} dP$$

$$= \int_\Omega e^1 \mathcal{E}^{\mathcal{B}_2}[t_{11} f^{2'} + t_{12} f^{2'}] dP + \int_\Omega e^2 \mathcal{E}^{\mathcal{B}_2}[t_{21} f^{1'} + t_{22} f^{2'}] dP$$

$$= \langle e, T' f' \rangle.$$

In this situation, conditional expectation is a bounded linear operator. Linearity is immediate and if $X \in L_\infty$ then $\sup|X| = \|X\| < \infty$, i.e., $\|X\| \geqq X(\omega) \geqq -\|X\|$; therefore, $\|X\| = \mathcal{E}^{\mathcal{B}}\|X\| \geqq \mathcal{E}^{\mathcal{B}} X \geqq -\mathcal{E}^{\mathcal{B}}\|X\| = -\|X\|$ so $\|\mathcal{E}^{\mathcal{B}} X\| \leqq \|X\|$; i.e., $\mathcal{E}^{\mathcal{B}}$ is a bounded operator. We have that if $T'f' = g$, $\|g^1\| = \|\mathcal{E}^{\mathcal{B}_1}[t_{11} f^{1'} + t_{12} f^{2'}]\| \leqq \|\mathcal{E}^{\mathcal{B}_1}\| \cdot M \cdot 2\|f'\| \leqq 2M\|f'\|$ and similarly for g^2. Therefore T' is well defined and continuous.

2) If $Y \subseteq F'$, then $T^{-1}(Y^\oplus) = [T'(Y)]^\oplus$, for $x \in T^{-1}(Y^\oplus) \Longleftrightarrow T(x) \in Y^\oplus \Longleftrightarrow \langle T(x), Y \rangle \geqq 0 \Longleftrightarrow \langle x, T'(Y) \rangle \geqq 0 \Longleftrightarrow x \in [T'(Y)]^\oplus$. Hence if Q is as given $Q^{\oplus\oplus} = Q$ by the comment following Lemma 1 and $[T^{-1}(Q)]^\oplus = [T^{-1}(Q^{\oplus\oplus})]^\oplus = [T'(Q^\oplus)]^{\oplus\oplus} = E$ closure of $T'(Q^\oplus)$ in E' because E' is as X in Lemma 1.

Applying the above two lemmas, Bratton's proof of the duality theorem follows verbatim where E, E', F, F', T are as given here and the weak* closure of $U'(P_R, P_E^\oplus, P_F^\oplus)$ is replaced by the E-closure of $U'(P_R, P_E^\oplus, P_F^\oplus)$ in E'.

Theorem: If U: $R \times E \to R \times E \times F$ is given by $U(\lambda, e) = (\lambda, e, \lambda b - eT)$ and $U'(P_R, P_E^\oplus, P_F^\oplus)$ is E closed in E', then the following are equivalent:

1) there exists a maximum of (e,a) for $e \geqq 0$, $eT \leqq b$;

2) there exists a minimum of (b,f') for $f' \geqq 0$, $T'f' \geqq a$;

3) $\Phi(e,f') = (e,a) + (b-eT,f')$ has a nonnegative saddle point.

Moreover, if there exists e,f' satisfying the constraints, then the problems 1, 2, 3 have solutions.

By making one additional assumption on the given problem a sufficient condition for applying the duality theorem is obtained.

<u>Assumption</u>: $(b^1(\omega),b^2(\omega)) > (\epsilon,\epsilon)$ for some $\epsilon > 0$ and for all $\omega \in \Omega$. This is slightly stronger than the admissibility of $(0,0)$.

We now proceed to show that the conditions of our modified duality theorem hold; i.e., if

$U: R \times E \to R \times E \times F$ is given by

$U(\lambda,e) = (\lambda,e,\lambda b-eT)$, then

$U'(P_R, P_E^{\oplus}, P_F^{\oplus})$ is E closed in E'.

U' is expressed as follows.

$$\begin{aligned}\langle(\lambda,e), U'(\delta,e',f')\rangle &= \langle U(\lambda,e), (\delta,e',f')\rangle \\ &= \lambda\delta + \langle e,e'\rangle + \langle \lambda b-eT,f'\rangle \\ &= \lambda(\delta + \langle b,f'\rangle) + \langle e,e' - T'f'\rangle \quad ;\end{aligned}$$

hence $U'\langle\delta,e',f'\rangle = (\delta + \langle b,f'\rangle,e'-T'f')$. Now if $\{(\delta_\alpha+\langle b,f_\alpha'\rangle, e_\alpha'-T'f_\alpha')\}$ is a sequence in $U'(P_R, P_E^{\oplus}, P_F^{\oplus})$ such that it E-converges, i.e., weakly, to some $(\bar{\mu},\bar{a})$, $\in R \times E'$. We wish to show that $(\bar{\mu},\bar{a}) \in U'(P_R, P_E^{\oplus}, P_F^{\oplus})$; i.e., if $\delta_\alpha \geqq 0$, $f_\alpha' \geqq 0$, $e_\alpha' \geqq 0$ and

$\delta_\alpha + \langle b,f_\alpha\rangle \to \bar{\mu}$, $e'_\alpha - T'f'_\alpha \to \bar{a}$ (weakly), then there exists $\bar{\delta} \geqq 0$, $\bar{e}' \geqq 0$, $\bar{f}' \geqq 0$ such that $\bar{\delta} + \langle b,\bar{f}'\rangle = \bar{\mu}$ and $\bar{e}'-T'\bar{f}' = \bar{a}$.

In the following, when subsequences are chosen they will return the original indices.

By assumption, $b \geqq \epsilon > 0$ so since $f'_\alpha \geqq 0$ therefore $\langle b,f'_\alpha\rangle \geqq 0$ and hence $\{\delta_\alpha\}$ and $\{\langle b,f'_\alpha\rangle\}$ are uniformly bounded and have convergent subsequences; i.e., $\delta_\alpha \to \bar{\delta}$, and we can make $\langle b,f'_\alpha\rangle$ converge monotonely to some K, such that $\bar{\delta} + K = \bar{\mu}$.

Now

$$\langle b,f'_\alpha\rangle = \int_\Omega b^1(\omega)f^{1'}_\alpha(\omega)dP + \int_\Omega b^2(\omega)f^{2'}_\alpha(\omega)dP$$
$$\geqq \epsilon\int_\Omega f^{1'}_\alpha(\omega)dP + \epsilon\int_\Omega f^{2'}_\alpha(\omega)dP$$

because $b(\omega) > (\epsilon,\epsilon)$ and $f'_\alpha \geqq 0$. From this inequality and the convergence of $\{\langle b,f'_\alpha\rangle\}$ we get that $\{\|f'_\alpha\|\}$ is bounded $[\epsilon\int_\Omega f^{1'}_\alpha(\omega)dP + \epsilon\int_\Omega f^{2'}_\alpha(\omega)dP \geqq \epsilon\max\{\|f^{1'}_\alpha\|,\|f^{2'}_\alpha\|\} = \|f'_\alpha\|]$, and that $\{f'_\alpha\}$ are pointwise bounded. Hence, by taking a Cantor diagonal, $\{f'_\alpha\}$ has a pointwise convergent subsequence; i.e., $f'_\alpha \xrightarrow{pt} \bar{f}'$. Now $\int \bar{f}^{1'}dP = \int \lim_\alpha f^{1'}_\alpha dP = \int \liminf_\alpha f^{1'}_\alpha dP \leqq \liminf \int f^{1'}_\alpha dP = \liminf \|f^{1'}_\alpha\| < \infty$ follows from the above and Fatou's lemma. Since $\bar{f}^{1'} \geqq 0$, $\|\bar{f}^{1'}\| < \infty$ and the same result on $\bar{f}^{2'}$ yields $\bar{f}' \in F'$. By monotonicity of $\langle b,\bar{f}'_\alpha\rangle \to K$, say, $\langle b,f'_\alpha\rangle - K \geqq 0$, hence

$$0 \leqq \int[\liminf(b\cdot f'_\alpha - K)]dP \leqq \liminf \int[b\cdot f'_\alpha - K]dP$$
$$= \lim[\int b\cdot f'_\alpha dP - K] = \lim[\langle b,f'_\alpha\rangle - K] = 0,$$

and

$$0 = \int[\liminf(b\cdot f'_\alpha - K)]dP = \int b\cdot\bar{f}'dP - K .$$

Therefore $\langle b,\bar{f}'\rangle = K$.

We have that $f'_\alpha \to \bar{f}'$ pointwise and $e'_\alpha - T'f'_\alpha \to \bar{a}$ weakly or, since E' is an L_1 space such that $P(\omega) > 0$ for all $\omega \in \Omega$, the convergence is also in the L_1 norm ([2], p. 259); i.e., $j=1,2$, $\lim \int |(e^{j'}_\alpha - [T'f'_\alpha]^j) - \bar{a}^j| \, dP = 0$. Now T and, therefore, T' are obviously pointwise continuous so $\lim_\alpha T'f'_\alpha(\omega) = T'\bar{f}'(\omega)$. Also weak convergence where $P(\omega) > 0$ and $\mathcal{A}$ discrete implies pointwise convergence [$\langle x,f\rangle = P(\omega)f(\omega)$ for an appropriate X ([2], p. 259)] so that pointwise $\lim_\alpha e'_\alpha = T'f'_\alpha + \bar{a} = \bar{e}$. Applying Fatou's lemma $j=1,2$, $0 \leqq \int |(e'_\alpha - [T'\bar{f}]^{j'}) - \bar{a}^j| \, dP =$

$= \int \liminf |(\bar{e}^j_\alpha - [T'f'_\alpha]^j) - \bar{a}^j| \, dP \leqq \liminf \int |(e'_\alpha - T'f'_\alpha)^j - \bar{a}^j| \, dP = 0;$

therefore $\bar{e}' = T'\bar{f} + \bar{a} \in E'$. Because e'_α, f'_α, $\delta_\alpha \geqq 0$, we have $\bar{e}' \geqq 0$, $\bar{f}' \geqq 0$, $\delta \geqq 0$ such that $\bar{\delta} + \langle b, \bar{f}' \rangle = \bar{\mu}$ and $\bar{e}' - T'\bar{f}' = \bar{a}$. Hence $(\bar{\mu}, \bar{a}) \in U'(P_R, P^{\oplus}_E, P^{\oplus}_F)$, and it is thence weakly (i.e., E) closed in E'.

So far we have established a sufficient condition for the equivalence statement of the duality theorem to hold.

The assumption of strict admissibility of zero is not sufficient to guarantee the existence of a maximum as is seen in the finite dimensional case (e.g., $T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $b = (2,3)$, $a = (1,1)$). As the last statement of the theorem indicates, a maximum exists if the constraint set is nonvoid and the dual problem has an admissible function.

We can derive a result analogous to the finite dimension situation which holds in the particular programming problem chosen. It is that if e_o is an admissible maximum then there exists a non-negative saddle point (e_o, f'_o), and, examining the Lagrange form, we get

$$0 = \langle b-e_oT, f_o \rangle = \int (b-e_oT)^1 f_o^{1'} dP + \int (b-e_oT)^2 f_o^{2'} dP \; ;$$

but the integrands are positive so $[e_oT(\omega)]^i = b^i(\omega)$ except possibly when $f^{i'}(\omega) = 0$, $i = 1, 2$.

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